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Generalized model of second order parametric filter

Abstract. *In this paper the method of determining response to any excitation of some class second order filters with non-periodically variable parameters has been proposed. The variability of the parameters has been described by series of eksponential functions. The restrictions of parametric functions which are guarantying correctness of this method have been presents. The results have been illustrated by the examples.*

Streszczenie. *W artykule została zaproponowana metoda wyznaczenia odpowiedzi na dowolne wymuszenie pewnej klasy filtrów o zmiennych w czasie parametrach drugiego rzędu z parametrami zmiennymi nieokresowo. Funkcje parametryzujące zostały opisane szeregiem funkcji eksponencjalnych. Przyjęte warunki przebiegu zmienności funkcji parametryzujących gwarantują poprawność stosowanej metody. Wyniki zostały zilustrowane przykładami.* ***(Uogólniony model filtru parametrycznego drugiego rzędu).***

Keywords: second order parametric system, LTV filter
Słowa kluczowe: układy parametryczne drugiego rzędu, filtry LTV

Introduction

Linear systems with time-varying parameters (LTV) are described by linear differential equations with variable coefficients [1], [2], [4]. The certain method of determination of these equations having good physical interpretation consists in variation of coefficients of differential equations corresponding to transfer functions of LLS (linear lumped stationary) systems [2]. In the case of a low-pass second order LTV filter the equation can be presented in the following form [3], [7]:

$$y''(t) + 2\sigma(t)\omega(t)y'(t) + \omega^2(t)y(t) = x(t) \quad (1)$$

where: $y(t)$ – filter response to excitation $x(t)$, $\omega(t)$ – time-varying angular frequency, $\sigma(t)$ – time-varying attenuation ratio.

In accordance with differential equations theory [3], [8] analytical solutions to equation (1) exist when the fundamental solutions to homogenous equation corresponding to equation (1) are known. Determination of fundamental solutions to such equations is possible only for special cases of parameters variability and requires knowledge about special functions of mathematical physics [3]. The mentioned above problem has been considered in earlier works [5], [7]. In the case of arbitrary non periodical functions $\omega(t)$, $\sigma(t)$ (compare with eq. (1)) the determination of fundamental solutions can be carried out using the method of successive approximations. The results obtained in this way are complex and difficult in interpretation.

In this article an analytical-numerical method of determination the fundamental solutions to equation (1) has been proposed [6]. The linear combination of obtained solutions can be interpreted as the LTV filter response to zero initial conditions with excitation $f(t)\equiv 0$. In the next step fundamental solutions the solution to non homogenous system (1) have been determined. The entire solution to equation (1) will represent the complete mathematical model of LTV filter described by parametric differential equation (1).

Formalization of analytical-numerical method

It has been assumed that functions $\omega(t)$ and $\sigma(t)$ representing the varying coefficients of equation (1) and further referred to simply as the parametric functions are expressed by formulae:

$$\omega(t) = \sum_{k=0}^{n_1} \omega_k \exp(-\alpha_k t), \alpha_0 = 0, \omega_0 > 0, \omega_k \in R, a_k \in R^+ , \quad (2)$$

$$\sigma(t) = \sum_{l=0}^{n_2} \sigma_l \exp(-\beta_l t), \beta_0 = 0, \sigma_0 > 0, \sigma_l \in R, \beta_l \in R^+ . \quad (3)$$

The functions defined above are expressed by the sum of functions with limited energy and constant additive components and it is representation of non-periodical functions with are reached the steady values ω_0 i σ_0 in the time $t \to \infty$. It can be proved [1], [5] that parametric filters described by (1) with earlier assumed conditions are asymptotic stable. After sufficiently long time the properties of LTV filters are identical as properties of low pass LLS filters with cut-off angular frequency ω_0 and attenuation ratio σ_0. One can expressed equation (1) in following form:

$$y''(t) + f(t)y'(t) + \left(f(t)g(t) - g^2(t) + g'(t)\right)y(t) = 0 , \quad (4)$$

thus:

$$f(t) = 2\omega(t)\sigma(t) , \quad (5)$$

$$f(t)g(t) - g^2(t) + g'(t) = \omega^2(t) , \quad (6)$$

where $g(t)$ is some unknown function. It can be proved [3], that particular solutions to equation (6) are expressed by formulae:

$$y_1(t) = \exp\left(-\int g(t)dt\right), \quad (7)$$

$$y_2(t) = \exp\left(-\int g(t)dt\right)\left(1 + \int \exp\left(-2\int\left[\omega(t)\sigma(t) - g(t)\right]dt\right)dt\right). \quad (8)$$

So the problem of solving equation (6) has been reduced to solving the Riccati equation in the following form:

$$g'(t) = g^2(t) - 2\omega(t)\sigma(t)g(t) + \omega^2(t) , \quad (9)$$

with function $g(t)$ as a variable. For earlier assumptions (compare equations (2) and (3)) the analytical solutions to Riccati equation are unknown [3], [8] and can be found only by using numerical methods.

Algorithm description

The block diagram of proposed method of analysis of second order LTV filter has been shown in figure 1. The algorithm has been implemented in MATLAB™.

First, Riccati equation (9) is solved for given input data using a numerical method. The solution is approximated by the exponential function series:

$$g(t) = \sum_{k=0}^{n_3} G_k \exp(-\gamma_k t), G_0 > 0, \gamma_k > 0, \gamma_0 = 0 . \quad (10)$$

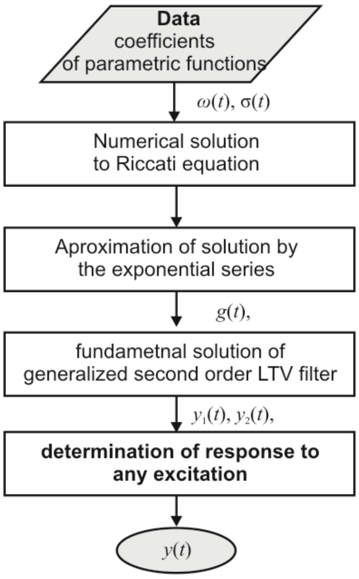

Fig 1. The analytical - numerical algorithm of analysis

Taking into consideration formulae (7) and (8) the fundamental solutions to equation (4) are determined:

(11) $$y_1(t) = \exp\left(G_0 t - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k t) \right),$$

(12)
$$y_2(t) = \exp\left(G_0 t - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k t) \right) \cdot$$
$$\cdot \int \exp[1 - 2[\omega_0 \sigma_0 t + \sum_{k,l=1}^{n_1+n_2} \frac{\omega_l \sigma_k}{\alpha_l + \beta_k} \exp(-(\alpha_l + \beta_k)t) +$$
$$- \omega_0 \sum_{k=1}^{n_2} \sigma_k \exp(-\beta_k t) - \sigma_0 \sum_{l=1}^{n_1} \omega_k \exp(-\alpha_l t) +$$
$$- G_0 t - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k t)]] dt.$$

The integral occurring in equation (12) should be computed using numerical methods or by series expansion of $\exp(\cdot)$ function.

In the second phase one can obtain the solution to second order differential equation (1) witch can be expressed as the system of two first order equations:

(13) $$\begin{bmatrix} p_1'(t) \\ p_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2(t) & -2\sigma(t)\omega(t) \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ x(t) \end{bmatrix}.$$

Taking into consideration the Wronski matrix (matrix of fundamental solutions and its derivates) [1]:

(14) $$\boldsymbol{W}(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_1'(t) \end{bmatrix},$$

one can determine the solution satisfies homogeneous system corresponding to the equation (1):

(15) $$\boldsymbol{p}(t) = \boldsymbol{p}_c(t) + \boldsymbol{p}_p(t),$$

where: $\boldsymbol{p}_c(t)$ - vector of complementary solutions:

(16) $$\boldsymbol{p}_c(t) = \boldsymbol{W}(t)\boldsymbol{W}^{-1}(t_0)\boldsymbol{x}(t_0),$$

$\boldsymbol{p}_p(t)$ - vector of particular solutions:

(17) $$\boldsymbol{p}_p(t) = \int_0^t \boldsymbol{W}(t)\boldsymbol{W}^{-1}(\tau)\boldsymbol{x}(\tau).$$

On the base of the fundamental solutions and Wronski matrix, witch is given by formula (14) and (18), one can calculate solution to non-homogenous equation (1) [3] Assuming zero initial conditions the solution takes the form:

(18) $$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \int_0^t \boldsymbol{W}(t)\boldsymbol{W}^{-1}(\tau)\boldsymbol{x}(\tau) d\tau.$$

Eventually the solution to equation (1) with zero initial conditions one can expressed as:

(19)
$$y(t) = A(t) \int_0^t [\exp\left(G_0 t - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k t) \right) \cdot$$
$$\cdot \int \exp[1 - 2[\omega_0 \sigma_0 t + \sum_{k,l=1}^{n_1+n_2} \frac{\omega_l \sigma_k}{\alpha_l + \beta_k} \exp(-(\alpha_l + \beta_k)t) +$$
$$- \omega_0 \sum_{k=1}^{n_2} \sigma_k \exp(-\beta_k t) - \sigma_0 \sum_{l=1}^{n_1} \omega_k \exp(-\alpha_l t) +$$
$$- G_0 t - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k t)]] dt \cdot \exp\left(G_0 \tau - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k \tau) \right) x(\tau) -$$
$$\exp\left(G_0 t - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k t) \right) \cdot \exp\left(G_0 \tau - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k \tau) \right) \cdot$$
$$\cdot \int \exp[1 - 2[\omega_0 \sigma_0 \tau + \sum_{k,l=1}^{n_1+n_2} \frac{\omega_l \sigma_k}{\alpha_l + \beta_k} \exp(-(\alpha_l + \beta_k)\tau) +$$
$$- \omega_0 \sum_{k=1}^{n_2} \sigma_k \exp(-\beta_k \tau) - \sigma_0 \sum_{l=1}^{n_1} \omega_k \exp(-\alpha_l \tau) +$$
$$- G_0 \tau - \sum_{k=1}^{n_3} \frac{G_k}{\gamma_k} \exp(-\gamma_k \tau)]] dt \cdot x(\tau)\, d\tau] d\tau,$$

where: $A(t)$ – determinant of Wronski matrix.

The formula (25) is a closed form of response of second order parametric filter with non-periodically variable parameters to any excitation $x(t)$.

Computing examples

In order to illustrate the method presented in the article the analysis of a second order LTV filters has been carried out. The three examples have been proposed. The variable parameters of filters have been described by formulae (2) and (3) with coefficient given by tables 1 and 2. The waveform of parametric functions have been presented on figures 2 and 3.

Table 1. Coefficients of functions $\omega(t)$

example		ω_0	ω_1	ω_2	α_1	α_2
I	$\omega_1(t)$	2	1	-1	5	1
II	$\omega_2(t)$	2	-0,5	2	1	3,5
III	$\omega_3(t)$	2	1	1	2	1,1

Table 2. Coefficients of functions $\sigma(t)$

example		σ_0	σ_1	σ_2	β_1	β_2
I	$\sigma_1(t)$	1	1,5	-1	5	1
II	$\sigma_2(t)$	1	2	-1	1	3,5
III	$\sigma_3(t)$	1	1,5	-2,5	2	1,1

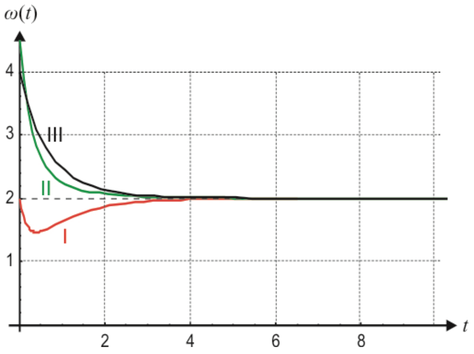

Fig 2. Examples of waveforms of functions $\omega(t)$

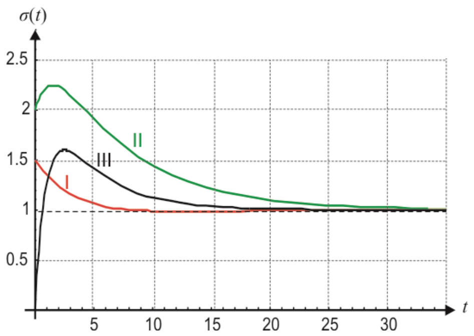

Fig 3. Examples of waveforms of functions $\sigma(t)$

The solution to Riccati equation (4) for assumed coefficients in case I and its approximation have been presented in figure 4.

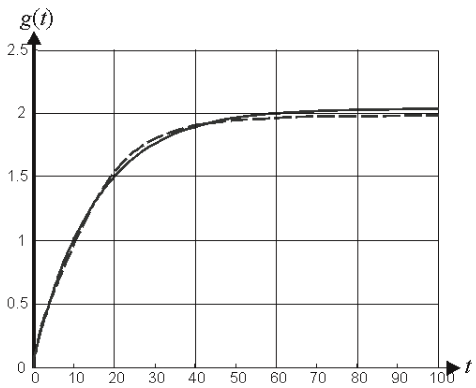

Fig 4. The solution to Riccati equation (dot line) and its approximation (solid line) for case I

Table 3. Coefficients of functions $g(t)$

przykład		g_0	g_1	g_2	γ_1	γ_2
I	$g_1(t)$	2,04	-0,66	-0,66	0,65	0,65
II	$g_2(t)$	1,92	-0,56	-0,56	0,05	0,05
III	$g_3(t)$	2,0	-0,4	-0,4	0,05	0,05

Numerical solution to equation (4) has been approximated by series of exponential functions (10). The coefficients of approximating formula are given also in table 3. Based on knowledge of function $g(t)$ one can find fundamental solution to equation (4) and one can determine the step response. The results of computing basing on equation (19), for zero initial conditions have been presented in figure 5

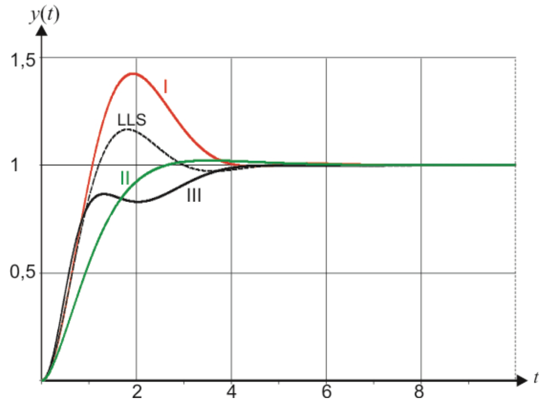

Fig 5. Examples of step responses of LTV filters with variable parameters.

For comparison of result the response of classical LLS filter with constant parameters $\sigma_0=1$, $\omega_0=2$ rad/s have been drafted also.

Conclusions

The proposed model of generalized parametric filter with non-periodically variable parameters allows to determination the system response to any excitation. Nevertheless, this method requires the numerical aided solving of an auxiliary Riccati equation.

This model will be used in the future for optimalization of dynamic properties of LTV systems and improving their characteristics comparing with LLS sections. The first and the second order sections are elementary blocks which enable building more complex filters with different properties. It allows to find optimal structures and determine values of parameters for minimum sitting time.

In the stationary state the sections with variable parameters are equivalent to LLS sections and the advantages of proposed approach consist in improvement of dynamic properties.

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